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LETTER TO THE EDITOR

On the susceptibility of the generalised square lattice Ising model

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Abstract. An exact functional relation is found for the susceptibility of the generalised square lattice Ising model. This result contains, as a particular case, a Fisher relation between the susceptibilities of the triangular and honeycomb lattices models. It is also shown that the closed-form expression for the susceptibility of the generalised square lattice, proposed by Syozi and Naya, satisfies the functional relation presented in this letter.

The zero-field magnetic susceptibility of the two-dimensional Ising model is not known, in explicit form, for any of the four regular lattices (square, triangular, honeycomb and Kagomé). For the anisotropic model, some special results have been obtained. Explicit expressions for the triangular lattice with multispin interactions and for the generalised square lattice, have been calculated on the disorder variety (Enting 1977, Dhar and Maillard 1985). On these varieties a dimensional reduction occurs and the model trivialises. Very recently (Debauche and Giacomini 1989) an explicit expression for the susceptibility of the anisotropic Kagomé lattice has been obtained when a relation between the three interactions parameters of the model is satisfied.

Also, in the critical region of the square lattice model, a great amount of information for the singular behaviour of the susceptibility has been obtained in recent years (for recent works and references to previous papers see Kong *et al* (1986) and Gartenhaus and McCullough (1988)).

With the aim of cumulating a set of exact results that could lead to its complete explicit determination, we present in this letter an exact functional relation for the susceptibility of the generalised square lattice (GSL) (or checkerboard lattice). This lattice contains the square, triangular and honeycomb lattices as particular cases. Let us now describe the procedure for obtaining this result.

Recently, Baxter (1986) has derived an exact functional relation for the partition function of the GSL Ising model with zero magnetic field. From this relation he has obtained expressions for the partition function and local correlations in terms of those of the regular square lattice Ising model. This functional relation can be generalised to the case where a magnetic field is present in one of the two sublattices of the GSL. From this result, as will be shown in the following, a functional relation for the susceptibility of the GSL can be derived.

Consider a square lattice L of $2N$ sites and periodic boundary conditions. Divide the edges into four classes $1, \ldots, 4$ as indicated in figure 1, and associate interaction coefficients $J_1, \ldots, J_4$ with the classes. With each site i associate a spin σ_i, with values $+1$ and -1. Let L' and L'' be the two sublattices of L, denoted by open and filled circles, respectively, in figure 1. With each site of sublattice L associate a magnetic field h. Then the partition function of this model is

$$Z = \sum_{\sigma} \exp\left\{\sum_{\langle ij\rangle} K_r\sigma_i\sigma_j + \sum_i H\sigma_i\right\} \tag{1}$$

where the inner sums are over all edges $\langle ij\rangle$ of L and all sites of L', respectively, r is the class of edge $\langle ij\rangle$, and the outer sum is over all values of $\sigma = \{\sigma_1, \ldots, \sigma_{2N}\}$. Here $K_r = J_r/k_B T$ and $H = h/k_B T$, where k_B is the Boltzmann constant and T the temperature. To obtain a functional relation for the partition function (1) we follow Baxter (1986). Let us sum over the N spins on sublattice L''. Then (1) becomes

$$Z = \sum_{\sigma'} \prod_{(ijkl)} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{2}$$

where the product is over all N faces of the lattice L' of broken lines in figure 1; i, j, k, l are the four sites around each such face, arranged as in figure 1. The sum is over the N spins on L' and

$$W(a, b, c, d) = 2\cosh(K_1 a + K_2 b + K_3 c + K_4 d)\exp\{(H/4)(a + b + c + d)\}. \tag{3}$$

Let us consider now the star–star relation (Baxter 1986)

$$2\cosh(K_1 a + K_2 b + K_3 c + K_4 d) = 2R\cosh(L_4 a + L_3 b + L_2 c + L_1 d)\exp\{M(bc - ad)\} \tag{4}$$

which is valid for all values ± 1 of the four spins a, b, c, d, if the parameters $L_1, \ldots, L_4$,

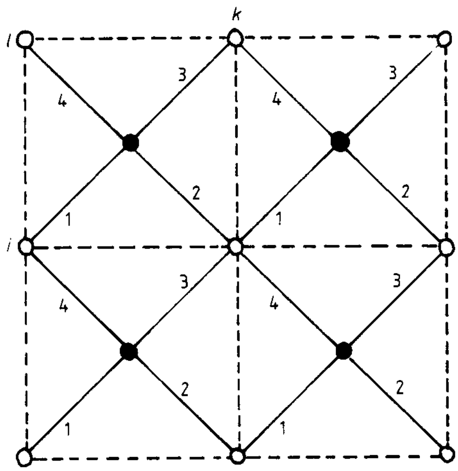

Figure 1. The generalised square lattice L (all circles), showing the sublattices L' (open circles) and L'' (filled circles). The four types of interactions $1, \ldots, 4$ are also indicated.

M and R satisfy the following equations:

$$\sinh(2L_i)\sinh(2K_i)=\Omega \qquad i=1,\dots,4 \tag{5a}$$

$$\begin{aligned}
\cosh(2L_1)&=\cosh(2K_2)\cosh(2P)-\coth(2K_1)\sinh(2K_2)\sinh(2P)\\
\cosh(2L_2)&=\cosh(2K_1)\cosh(2P)-\coth(2K_2)\sinh(2K_1)\sinh(2P)\\
\cosh(2L_3)&=\cosh(2K_4)\cosh(2P)+\coth(2K_3)\sinh(2K_4)\sinh(2P)\\
\cosh(2L_4)&=\cosh(2K_3)\cosh(2P)+\coth(2K_4)\sinh(2K_3)\sinh(2P)
\end{aligned} \tag{5b}$$

$$R=\prod_{i=1}^{4}\left(\frac{\sinh(2K_i)}{\sinh(2L_i)}\right)^{1/4} \tag{5c}$$

$$\tanh(2M)=\frac{\sinh(2K_2)\sinh(2K_3)-\sinh(2K_1)\sinh(2K_4)}{\cosh(2K_2)\cosh(2K_3)+\cosh(2K_1)\cosh(2K_4)} \tag{5d}$$

with

$$\tanh(2P)=\frac{\sinh(2K_1)\sinh(2K_2)-\sinh(2K_3)\sinh(2K_4)}{\cosh(2K_1)\cosh(2K_2)+\cosh(2K_3)\cosh(2K_4)} \tag{5e}$$

$$\Omega^2=\prod_{i=1}^{4}\frac{\sinh(2K_i)\cosh(K_1^*+K_2^*+K_3^*+K_4^*+2K_i^*}{\sinh(2K_i^*)\cosh(K_1+K_2+K_3+K_4-2K_i)} \tag{5f}$$

and

$$\exp(-2K_i^*)=\tanh(K_i). \tag{5g}$$

Now, we can replace the factor $2\cosh(K_1a+\dots+K_4d)$ in (3) by the right-hand side of (4). But the factor $\exp[M(bc-ad)]$ is cancelled when the product of the Boltzmann weights $W(a, b, c, d)$ is performed over all faces of L'. Therefore, taking into account (2), we have the following functional relation for the partition function (1):

$$Z(H, K_1, K_2, K_3, K_4)=R^N Z(H, L_4, L_3, L_2, L_1) \tag{6}$$

with L_i and R given by equations (5). Equation (6) is valid for arbitrary values of magnetic field H and, as has been shown above, the relations between parameters L_i and K_i are independent of H. Therefore, the following result can be established for the zero-field susceptibility of sublattice L':

$$\chi_0^{(L')}(K_1, K_2, K_3, K_4)=\chi_0^{(L')}(L_4, L_3, L_2, L_1) \tag{7}$$

where

$$\chi_0^{(L')}=\frac{1}{k_B T}\frac{\partial^2}{\partial H^2}\frac{1}{2N}\log Z(H, K_1, K_2, K_3, K_4)\bigg|_{H=0}. \tag{8}$$

As has been shown by Fisher (1959), a relation can be established between the susceptibility of a sublattice L' of a bipartite lattice L and the total susceptibility χ_0 of L. In our case, this relation is as follows:

$$\chi_0^{(L')}(K_1, K_2, K_3, K_4)=\tfrac{1}{4}[\chi_0(K_1, K_2, K_3, K_4)+\chi_0(-K_1, -K_2, -K_3, -K_4)]. \tag{9}$$

By using (7) and (9), we finally obtain the desired functional relation for the total susceptibility χ_0:

$$\begin{aligned}
&\chi_0(K_1, K_2, K_3, K_4)+\chi_0(-K_1, -K_2, -K_3, -K_4)\\
&\qquad=\chi_0(L_4, L_3, L_2, L_1)+\chi_0(-L_4, -L_3, -L_2, -L_1)
\end{aligned} \tag{10}$$

where parameters L_i are given in terms of K_i by means of equations (5). As can be easily seen from these equations, the transformation T that leads from parameters K_i to parameters L_i is involutive, that is to say $T^2 = \mathsf{I}$, where I is the identity transformation.

For the special case of the anisotropic square lattice ($K_1 = K_3$ and $K_2 = K_4$), transformation T trivialises, and we have $L_4, L_3, L_2, L_1 = K_3, K_4, K_1, K_2$. Equation (10) represents for this case a simple geometrical symmetry of the model.

The parameter Ω^2, given in (5*f*), is the relevant (temperature-like) variable of the model. Varying $K_1, \ldots, K_4$, while keeping Ω^2 fixed, does not affect the phase of the system: if it is ordered (disordered) for one such physical set of values of $K_1, \ldots, K_4$, then it is ordered (disordered) for all. For $\Omega^2 > 1$ the system is ordered, for $\Omega^2 < 1$ it is disordered and $\Omega^2 = 1$ determines the critical variety of the model. As can be easily seen from equations (5*a*), (5*f*) and (5*g*), Ω^2 is invariant under the transformation T and under the reversal of the sign of the four interactions K_i; that is to say

$$\Omega^2(K_1, \ldots, K_4) = \Omega^2(L_4, \ldots, L_1) = \Omega^2(-K_1, \ldots, -K_4). \tag{11}$$

Then, the four terms of equation (10) have the same value of Ω^2.

Let us consider the special case $K_4 = +\infty$. The GSL Ising model is equivalent, in this limit, to the anisotropic triangular Ising model with N sites and interaction coefficients K_1, K_2, K_3. Moreover, the resulting magnetic field on the triangular lattice is twice the field of the GSL. Therefore, we have the following relation between the susceptibilities of the GSL and triangular lattice models:

$$\chi_{0\,\mathrm{GSL}}(K_1, K_2, K_3, K_4 = +\infty) = 2\chi_{0\,\mathrm{TRIANG}}(K_1, K_2, K_3). \tag{12}$$

In the left-hand side of (10) we have also, in this limit, the term $\chi_0(-K_1, -K_2, -K_3, -K_4 = -\infty)$. When $K_4 = -\infty$, spins connected by this interaction must be opposed. This leads to a cancellation of the magnetic field, when it acts on all sites of the lattice (as is the case for calculating the total susceptibility χ_0). Therefore we have, in particular,

$$\chi_{0\,\mathrm{GSL}}(-K_1, -K_2, -K_3, -K_4 = -\infty) = 0. \tag{13}$$

On the other hand, when $K_4 = +\infty$, it can be deduced from equations (5) that (Baxter and Choy 1988)

$$\begin{aligned} &L_4 = 0 \\ &\sinh(2K_i)\sinh(2L_i) = \Omega \qquad i = 1, 2, 3 \\ &\cosh(2L_i) = \cosh(2K_j)\cosh(2K_k) + \coth(2K_i)\sinh(2K_j)\sinh(2K_k) \end{aligned} \tag{14a}$$

for all permutations (i, j, k) of $(1, 2, 3)$, with

$$\Omega^2 = \frac{16(1 + v_1v_2v_3)(v_1 + v_2v_3)(v_2 + v_3v_1)(v_3 + v_1v_2)}{(1 + v_1^2)^2(1 - v_2^2)^2(1 - v_3^2)^2} \tag{14b}$$

and $v_i = \tanh(K_i)$, $i = 1, 2, 3$.

For this particular case, the parameters L_1, L_2, L_3 are obtained from K_1, K_2, K_3 by means of the well known star-triangle relation, defined by equations (14). The GSL Ising model with interaction coefficients $L_4 = 0$, L_3, L_2, L_1, and $2N$ sites is equivalent to the anisotropic honeycomb lattice Ising model with interaction parameters L_1, L_2, L_3, and $2N$ sites. Therefore we have

$$\chi_{0\,\mathrm{GSL}}(L_4 = 0, L_3, L_2, L_1) = \chi_{0\,\mathrm{HONEY}}(L_1, L_2, L_3). \tag{15}$$

In consequence, for the particular case $K_4 = +\infty$, relation (10) becomes

$$\chi_{0\,\mathrm{TRIANG}}(K_1, K_2, K_3) = \tfrac{1}{2}(\chi_{0\,\mathrm{HONEY}}(L_1, L_2, L_3) + \chi_{0\,\mathrm{HONEY}}(-L_1, -L_2, -L_3)) \tag{16}$$

i.e. the well known relation between the susceptibilities of the triangular and honeycomb lattices models derived by Fisher (1959).

Let us return now to the general case. The susceptibility of the GSL model also satisfies the so-called inversion relation:

$$\chi_0(K_1, K_2, K_3, K_4) + \chi_0(K_1 + \mathrm{i}\pi/2, -K_2, K_3 + \mathrm{i}\pi/2, -K_4) \tag{17}$$

where the second term must be considered as the analytical continuation of the first term (for a review of the inversion relation, see Maillard (1985)). From equation (17), geometrical symmetries of the model and the disorder solution, severe constraints are obtained on the resummed temperature expansions. However, these constraints are not sufficient to completely determine χ_0. The functional relation (10) presented in this letter represents a new piece of exact information to be imposed to resummed temperature expansions. It would be interesting to study the constraints imposed by this relation.

It is worth pointing out that using symmetry properties of the Ising model (star-triangle relation, duality transformation, geometrical symmetries), the one-dimensional limits and the anisotropic high-temperature expansions, Syozi and Naya (1960) proposed a closed expression for the susceptibility of the GSL given by

$$\chi_0 = \frac{1}{k_B T} \frac{(\Sigma_{i=1}^4 S_i^2 + 2C_1C_2C_3C_4 + 2S_1S_2S_3S_4 + 2)^{1/2} + \Sigma_{i=1}^4 S_i}{C_1C_2C_3C_4 + S_1S_2S_3S_4 + 1 - \Sigma_{i<j} S_iS_j} (1 - \Omega^2)^{1/4} \tag{18}$$

where $S_i = \sinh(2K_i)$, $C_i = \cosh(2K_i)$ with $i = 1, \ldots, 4$.

This closed (approximated) expression is singular on the critical variety of the model and has actually the correct critical exponent $\gamma = \frac{7}{4}$. Moreover, this expression satisfies the inversion relation (17) and reduces to the result of Dhar and Maillard (1985) on the disorder variety (Hansel and Maillard 1987). It can be proved from equations (5) that (18) satisfies the functional relation (10). Hence this remarkable expression for χ_0 satisfies all known exact results, in spite of the fact that it is not exact. It would be interesting to find the lattice model for which (18) is the exact susceptibility, and to compare it with the usual Ising model.

Finally, let us remark that from relation (6) with $H = 0$, Baxter (1986) has obtained several interesting results for the GSL Ising model. Can some of these results be generalised to the case of non-zero field studied in this letter?

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